F. More on Measurements

(i) When could we predict outcome with 100% certainty?

Use 1D Box as example [but results are general]

(1D Box: there is a Hamiltonian [total energy] $\hat{H}$)

Given the state is $\psi_5(x)$ [eigenfunction of $\hat{H}$: $\hat{H}\psi_5 = E_5\psi_5$]

Consider measurements on Energy (e.g. 1M copies, each in ψ_5)

$$\langle H \rangle = \int_{-\infty}^{\infty} \psi_5^*(x)\, \underbrace{\hat{H}\, \psi_5(x)}\, dx \quad \text{[by formula]}$$

OR $\langle E \rangle$

$$= \int_{-\infty}^{\infty} \psi_5^*(x)\, E_5 \psi_5(x)\, dx \quad [\psi_5 \text{ is eigenfunction of eigenvalue } E_5]$$

$$= E_5 \int_{-\infty}^{\infty} |\psi_5(x)|^2 dx = E_5 \quad [\psi_5(x) \text{ is normalized}]$$

$\langle H \rangle$ is the expectation value of energy for the state ψ_5

$$(\Delta H)^2 = \langle H^2 \rangle - \langle H \rangle^2 = \int_{-\infty}^{\infty} \psi_5^* \hat{H} \underbrace{\hat{H} \psi_5}_{E_5 \psi_5} dx - E_5^2$$

$$= E_5 \int_{-\infty}^{\infty} \psi_5^* \underbrace{\hat{H} \psi_5}_{E_5 \psi_5} dx - E_5^2$$

$$= E_5^2 - E_5^2 = 0$$ (No spread in results!)

Meaning: Given ψ_5 and measure energy, the answer is E_5 with 100% certainty

[e.g. 1M data are all E_5]

Generalization:

- Measure quantity A and given state ϕ_i is eigenfunction of $\hat{A}$

  Result is 100% certain to be a_i (eigenvalue of ϕ_i)

  i.e. $\langle A \rangle = \int \phi_i^* \hat{A} \phi_i \, dx = a_i \int \phi_i^* \phi_i \, dx = a_i$

  $$(\Delta A)^2 = \int \phi_i^* \hat{A}\hat{A} \phi_i \, dx - a_i^2 = 0$$

(ii) If $\Phi(x)$ is NOT an eigenfunction, what can be predict?

Example: Measure energy and state is

$$\Phi(x) = c_1 \psi_1(x) + c_5 \psi_5(x)$$

$\psi_1(x) \leftrightarrow E_1$

$\psi_5(x) \leftrightarrow E_5$

<u>Mean energy</u>

$$\langle E \rangle = \langle H \rangle = \int_{-\infty}^{\infty} (c_1^* \psi_1^* + c_5^* \psi_5^*) \hat{H} (c_1 \psi_1 + c_5 \psi_5) dx \quad \text{(by formula)}$$

$$= \underbrace{|c_1|^2}_{} E_1 + \underbrace{|c_5|^2}_{} E_5 \quad (\hat{H}\psi_n = E_n \psi_n \text{ and orthonormal})$$

Interpretation: prob. of getting E_1 ; prob. of getting E_2

$|c_n|^2 = 0$ for $n = 2, 3, 4, 6, 7, \ldots$

(zero prob. of getting $E_2, E_3, E_4, E_6, E_7, \ldots$)

Inspect Math Approach

Measure "energy" $\Rightarrow$ operator $\hat{H}$ $\Rightarrow$ $\hat{H}\psi_n = E_n\psi_n$ (must solve TISE first)

Given <u>any state</u> $\Phi(x)$:

<u>Step 1</u>: Express $\Phi(x) = c_1\psi_1 + c_2\psi_2 + \cdots = \sum_{n=1}^{\infty} c_n\psi_n$

This can always be done, as $c_n = \int_{-\infty}^{\infty} \psi_n^*(x)\,\Phi(x)\,dx$

<u>Step 2</u>: Read out coefficients

$|c_n|^2$ = Probability of getting E_n in measurements

Generalization

Measure "A" $\Rightarrow$ operator $\hat{A}$ $\Rightarrow$ $\hat{A}\phi_i = a_i \phi_i$ (Must solve eigenvalue problem of $\hat{A}$ first)

Given any state $\Phi(x)$:

Step 1: Express $\Phi(x) = \tilde{c}_1 \phi_1 + \tilde{c}_2 \phi_2 + \cdots = \sum_{n=1}^{\infty} \tilde{c}_n \phi_n$

This can always be done, as $\tilde{c}_n = \int_{-\infty}^{\infty} \phi_n^*(x)\, \Phi(x)\, dx$

Step 2: Read out coefficients

$|\tilde{c}_n|^2 =$ Probability of getting a_i in measurements

Remark: The content on this page can be taken as a QM postulate. If so, then $\langle A \rangle = \int \Phi^* \hat{A} \Phi \, dx$ follows.